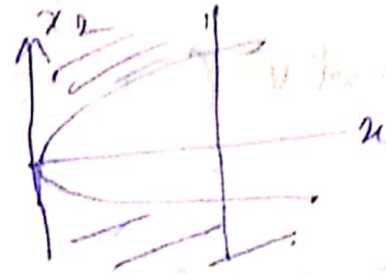
Ex: Prove that in $\mathbb{R}^2$, $X_1 = \{(x_1, x_2) \mid x_2^2 \geq 4x_1\}$ and $X_2 = \{(x_1, x_2) \mid x_1 x_2 \leq 4\}$ are not convex sets while the set $X_3 = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + x_3 \leq 4\}$ is a convex set.

Sol: (i) Con. $X = \{(x_1, x_2) \mid x_2^2 \geq 4x_1\}$



$(1, 3) \in X$ as $9 > 4$

$(1, -3) \in X$ as $9 > 4$

But for $\lambda = 0.5$

$\lambda (1, 3) + (1 - \lambda)(1, -3)$

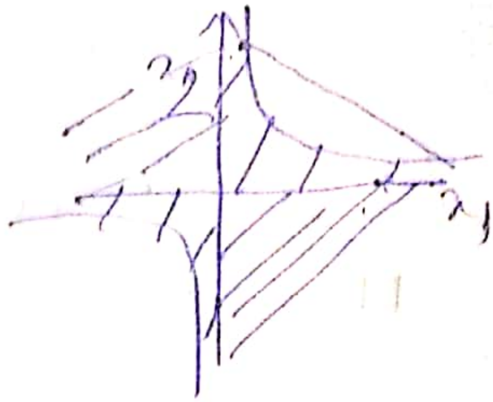
$= 0.5(1, 3) + 0.5(1, -3)$

$= (1, 0)$

we have $0^2 \not> 4$

Hence X_1 is not convex set

$X_2 = \{(x_1, x_2) \mid x_1 x_2 \leq 4\}$



~~(2, 1) ∈ X_2~~

~~(-1, -2) ∈ X_2~~

for $\lambda = 0.5$

We notice that $(30,0) \in X_2$ and $(0,30) \in X_2$.

But for $\lambda = 0.5$ there convex combination

$$\cancel{\lambda(3,0) + (1-\lambda)(0,3) = 0.5(3,0)}$$

$$\lambda(30,0) + (1-\lambda)(0,30) = 0.5(30,0) + 0.5(0,30)$$

$$= (15,0) + (0,15) = (15,15) \notin X_2$$

$[\because 225 > 4]$

Hence X_2 is not convex.

$$X_3 = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + x_3 \leq 4\}$$

Let $(x_1, x_2, x_3) \in X_3$, $(y_1, y_2, y_3) \in X_3$

$$\cancel{2x_1 - x_2 + x_3 \leq 4, y} \quad 2x_1 - x_2 + x_3 \leq 4$$

$$2y_1 - y_2 + y_3 \leq 4$$

Now their convex combination is

$$\cancel{\lambda(2x_1 - x_2 + x_3) + (1-\lambda)(2y_1}$$

$$\lambda(x_1, x_2, x_3) + (1-\lambda)(y_1, y_2, y_3) \text{ for } 0 \leq \lambda \leq 1$$

$$= (\lambda x_1 + (1-\lambda) y_1, \lambda x_2 + (1-\lambda) y_2, \lambda x_3 + (1-\lambda) y_3)$$

$$2\{\lambda x_1 + (1-\lambda) y_1\} - (\lambda x_2 + (1-\lambda) y_2) + (\lambda x_3 + (1-\lambda) y_3)$$

$$= \lambda(2x_1 - x_2 + x_3) + (1-\lambda)(2y_1 - y_2 + y_3)$$

$$\leq 4\lambda + (1-\lambda)4 = 4$$

So ~~their~~ their convex combination belongs to X_3. Hence X_3 is convex